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Phase Space, Fibre Bundles and Current Algebras

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Abstract

The purpose of this paper is to extend into phase space the cellular description introduced by Bohm *et al.* (1970) and to show how this may help to give an understanding of the current algebra approach to elementary particle phenomena. We investigate this cellular structure in phase space in some detail and show how certain features of the structure may be described in terms of the mathematics of fibre bundle theory. The frame bundle is discussed and compared with the Yang-Mills theory. As a result of this discussion we are able to introduce generalised currents which are related to the duals of the curvature forms, and these are shown to span the Lie algebra of a sub-group of the structure group of the frame bundle. We then discuss the implications of these results in terms of our cell structure. By assuming that the de Rahm cohomology, defined by the curvature forms and their duals, reflect a cohomology on the integers defined on the original cell structure, we show that the currents and 'curvature' can be given a meaning in terms of a discrete structure. In this case the currents only span a Lie algebra in some suitable limit, implying that a description using Lie algebras is only an approximation.

1. *Introduction*

In a recent paper (Bohm *et al.*, 1970) we have suggested that it is possible to find new descriptions in physics which have many features similar to those contained in quantum theory (e.g. discreteness, superposition, potentialities, etc.) without using the Hilbert space formalism and its probability interpretation. To achieve this, we have suggested that the classical notion of an object in itself is not to be taken as a *basic* descriptive term. This does not deny its relevance in, for example, the kinetic theory, but here it is to be regarded as an abstraction from something more basic.

From a careful consideration of general questions concerning our primitive perceptions, and from a careful consideration of the implicit notions contained in quantum theory (in particular, the consequences of the indivisibility of the quantum of action), we have suggested that primary relevance should be given to activity and wholeness in the sense of undivided movement. In terms of these notions, the apparatus itself is to be regarded as being an aspect of the movement so that there is no longer a sharp distinction between the apparatus and the observed phenomena. Each experimental arrangement potentiates a content with which physics is concerned in an essential way.

When primary relevance is given to activity and wholeness, the traditional mode of description using particles and fields together with the space-time continuum can no longer be regarded as basic. We have already suggested (Hiley, 1968; Bohm *et al.*, 1970) that the appropriate mathematical description should use the descriptive terms of homology and cohomology theory (e.g. simplexes, complexes, chains, cochains, etc.) that is, the basic description should be cellular.

In this paper we wish to relate this cellular description to some attempts that have recently been made to understand elementary particle phenomena. In particular we wish to discuss some aspects of the bilocal theory proposed by Yukawa (1965) and extended by Takabayasi (1965, 1970). Their theory can be regarded as imposing a cellular structure on *space-time*. We point out that such a structure is arbitrary and not supported by any direct experimental evidence. On the other hand, as we have pointed out elsewhere (see Bohm, 1969; Hiley, 1968), quantum phenomena call for a description based on a cell structure in *phase space* and we propose to investigate some aspects of this structure.

In its most general form our theory replaces phase space by an abstract simplicial complex, together with a cohomology with values in the integers, thus giving a discrete structural description. However, in order to make rapid contact with present theories, we assume our simplexes are homeomorphic to Euclidean simplexes. For these simplexes we show how certain features of the complex can be described approximately by means of fibre bundle theory using, in particular, the frame bundle. This mathematical theory contains formal features that are similar to those used in the Yang-Mills theory (see Utiyama, 1956) and this enables us to give a structural meaning to gauge invariance and current algebras.

Although the similarity between the frame bundle theory and the Yang-Mills theory has already been noted by others (e.g. Lubkin, 1963; Herman, 1966) our approach is different in several essential ways. We do not start with a Lagrangian field theory which is made gauge invariant. As a result we do not regard the description as one in which fields are considered to be in interaction. To us, analysis into parts is not relevant and the interaction is to be regarded as inseparable from the overall structure, much in the same vein as in general relativity where the forces are described by the geometry. However, our structure is not a continuum metric geometry, but a cellular structure which is basically discrete. The frames that are used in the frame bundle give an approximate description of certain features of this structure. For example, gauge invariance implies that it is the relative orientation and not the absolute orientation of the frames that is an important feature of the structure.

The form in which we develop the mathematics is also different from the earlier theories. We use differential forms throughout and interpret these forms as defining a de Rahm cohomology. We then assume that this de Rahm cohomology reflects a cohomology with values in the integers that can be defined on the basic simplicial complex. The relevant formulæ used to describe the frame bundle can then be given a meaning in terms of the discrete structure.

In Section 2 we show how the work of Yukawa and Takabayasi enables us to use fibre bundle theory to describe certain features of the cell structure. In Section 3 we briefly review some of the pertinent features of fibre bundle theory and develop the relevant formulæ required to discuss the formal relationship between the bundle theory and the Yang-Mills theory. This relationship is then discussed in Section 4. In this section we also indicate why we consider the bundle theory to be more appropriate from the point of view that we are adopting. In Section 5 we use this theory to give a structural meaning to the currents and show that the current commutation relations emerge as a natural consequence of the bundle description. Finally, we point out that when the full consequences of the cell structure are taken into consideration, the current commutation relations no longer span a Lie algebra, indicating that a description in terms of a Lie algebra is only an approximation.

2. *Cellular Descriptions and the Multilocal Model*

Cellular descriptions have, of course, already been proposed for various reasons (Ishiwara, 1915; Wheeler, 1963) but we wish to make particular reference to the work of Yukawa (1965) and Takabayasi (1965, 1970) which contain certain basic features pertinent to our discussion. Yukawa's motivation for introducing a cellular description stems from the divergence difficulties in quantum field theory which he attributes to the neglect of the finite size of the elementary particles. If the particles do have a finite size, and experimental scattering data clearly indicate that they appear to behave like extended structures in space-time (Olsen *et al.*, 1961), then one way of discussing this structure is to assume that the free-fields used to describe the particles can only be specified over finite regions in space-time. (This, in itself, is different from the usual method which uses *local* fields and form factors.)

If Yukawa's theory is not to reduce to a local field theory under some transformation then we must assume that the domains can never be reduced to a point. Thus, it is necessary to impose some restriction on the size and the shape of the domains. The simplest idea that Yukawa considered was to assume that these domains were spherical with a radius l_0, that is, a fundamental length is introduced. This feature of the bilocal theory is unsatisfactory because the shape and the size of the domains are arbitrarily imposed and there is, as yet, no evidence to support the notions of a fundamental length in space-time.

As indicated earlier, instead of considering a cellular structure in a space-time manifold, we consider such a structure in a phase space manifold (i.e. the kinematic properties rather than the static properties are considered). In this case it is known from quantum theory that phase space has a natural cellular structure, the volume of the cells being determined by Planck's constant. Furthermore, it has already been pointed out (Bohm, 1969; Hiley, 1968) that the shape of the cells depend on the experimental conditions, thus showing that neither the shape nor the size of the cells are arbitrarily imposed from outside. In fact the overall cell structure is inseparable from the experimental conditions which lead to the kind of wholeness suggested by quantum theory. Thus we argue that if a cellular description is to be regarded as basic, then this structure arises naturally in the phase space rather than in space-time. In other words, when the cells (or simplexes) are homeomorphic to Euclidean cells (or simplexes), the appropriate manifold will be a phase space manifold rather than a space-time manifold.

We must next question how the concept of a particle is to enter the description. In the Yukawa theory, the particle is an object in itself which 'jumps' from cell to cell. This idea is, however, not consistent with the notions we are discussing here. For, in our point of view, the particle must be abstracted from the invariant features of the whole movement† and cannot be thought of as being an independently existing entity. Indeed, the cells must be used to describe the invariant features of the movement and these invariant features must be combined to give an explanation of the properties of 'classical particles'. (We will see that the invariant features can be regarded as 'currents'.)

In order to show how such a connection can be made, we will outline the main steps used in the mathematical description. As we have already pointed out, we are assuming that the simplexes are homeomorphic to Euclidean simplexes. This approximation is equivalent to covering a manifold with a simplicial complex, and we want to show how the manifold can be used to take into account the cell structure. To do this we, in fact, make use of the ideas proposed by Yukawa (1965) and generalised by Takabayasi (1965, 1970). In its simplest form, the spherical cells could be specified by a point x^μ, taken to be the centre of the sphere, together with a radius vector r^μ. In the modified, more general version, the cells can undergo linear deformation and to describe this deformation, one needs a mean position x^μ, together with a frame $X_\lambda{}^\mu$. If the cells are embedded in a four-dimensional manifold, then a four-dimensional frame must be used. If more complicated deformations are necessary, then linearity can be preserved by considering a frame with higher dimensionality.‡ Trans-

† The whole movement is called the holomovement in Bohm *et al.* (1970).

‡ A similar approach is used in the fluid droplet model in which higher multipole moments are treated as linear.

formations among the sets of equivalent frames of a given dimensionality will, evidently, be through the general linear group, or one of its sub-groups. These sub-groups will arise if there is a greater symmetry in the cells. Hence the group, which we will call the structure group, describes the symmetry properties of the cells.

Thus we have argued that if a cell structure can be embedded in a manifold, the cell description can be replaced by a frame description; the symmetry properties of the cells can be described by a set of frames related to each other at each point of the phase space through some suitable group. However, in using the manifold description, one of the important features of quantum theory, namely, discreteness has already been lost and therefore it is not surprising that the resulting description will contain features that are similar to unquantised classical theories.

Furthermore, in the approximation in which the cell description is replaced by the frame description, it is apparent that the absolute size and absolute 'orientation' of the cells is not relevant. The frames themselves refer only to relative size and to relative 'orientation' of neighbouring cells.

We thus begin to see emerging a natural meaning to gauge invariance which is similar to the original idea introduced by Weyl (1922). In order to show this more clearly and to relate it to the conventional meaning (i.e. invariance under arbitrary changes in the phase of the wave function) we first call attention to a certain well-known mathematical structure, comprising a manifold, a set of frames at each point of the manifold, and a structure group. This is, in fact, called a frame bundle which is, in turn, a particular example of a more general mathematical structure called a principal fibre bundle. Although this structure is well known in mathematics, it has not been used much in physics, and only recently has it become important in the theory of group representations (see Herman, 1966). Thus its power for discussing basic geometric and topological questions in physics has hardly been appreciated (but see Mackey, 1963; Mayer, 1966).

We note further that there is a very close connection between the fibre bundle and the Yang-Mills theory. This connection has already been briefly referred to by Herman (1966). But it was Lubkin (1963) who first showed the connection explicitly when he attempted to give a geometric meaning to gauge transformations. However, he does not use the full implications of the fibre bundle theory and stayed close to conventional field theory without attempting to change any basic concepts. On the other hand, to us the frame bundle is a way to obtain an approximate description of a cell structure. This approximation will be particularly important in describing features of the structure which depend strongly on the relative 'orientation' of cells. The cell structure is inseparable from the apparatus and, hence, the relative 'orientation' of the cells is not arbitrary. The way the cells fit together is the description we require and the fibre bundle is one way to obtain an approximate description of this structure.

3.1. *Frame Bundles*

As the theory of fibre bundles is unfamiliar to most physicists, we would like to present a somewhat simplified and intuitive account of the theory confining our discussion to those aspects that will be of direct relevance to this paper. No attempt will be made to discuss the formal properties of the structure in any detail as adequate discussions already exist elsewhere. (e.g. Bishop & Crittenden, 1964.)

Consider a set of frames at each point m of an n-dimensional manifold M, which we will call the base space. By frame, we mean a set $(\mathbf{e}_1, \ldots \mathbf{e}_n)$ of linearly independent tangent vectors. These frames can be thought of as forming the basis of a vector space, F (called the fibre), so that each point on M has a fibre associated with it. We can generalise further by considering a super-space, P formed from the union of the fibres and the manifold, M. We call this new space the bundle space. It is the manifold which is locally of the form $M \times F$, i.e. the local structure is 'trivial'.

Each point $p \in P$ is associated with a unique point m on M, that is, we can introduce a mapping $\pi : P \to M$ such that $\pi(p) = m$. The inverse of this mapping, $\pi^{-1}(m)$, picks out all points on the same fibre, F, i.e. $\pi^{-1}(m) \in F$. Thus, in terms of the frame bundle, π associates with the origin of each frame a point $m \in M$ and $\pi^{-1}(m)$, gives the set of frames at m.

As we have already indicated, the set of frames at a point m are related through a group G which, in this particular case, is $GL(n, R)$ or one of its subgroups. Since the frames are regarded as the bases of the vector space F, we see that G acts on F and, since F is in P, G also acts on P in the following manner. If $p \in F$, then every other p' on the same fibre is given by $p' = pg$ with $g \in G$. This group is called the structure group. In the case of the frame bundle, we can write $p = (m, \mathbf{e})$ and $p' = (m, \mathbf{f})$ so that

$$(m, \mathbf{f}) = (m, \mathbf{e}g) \tag{3.1}$$

In the structure developed thus far there is no way of introducing the idea of a 'geometric object', i.e. what physicists call tensors. Normally these are introduced in terms of a coordinate system, but so far we have not introduced coordinates into the fibre bundle. In fact, the power of the bundle theory is that the structures can be described without the need for a coordinate representation.

Now to see how tensors are to be introduced, first consider the intrinsic definition of a vector, $\mathbf{V}$, written as the product $v^i \mathbf{e}_i$ (summation over repeated indices). Under the action of $g \in G$, the set $\{\mathbf{e}_i\}$ transform to the right by equation (3.1) and so, if $\mathbf{V}$ is to remain unchanged under each $g \in G$ (i.e. it is to be a 'geometric object'), the set $\{v^i\}$ must be transformed on the left by g^{-1}.

To describe this action, in general, we introduce a vector space $\mathscr{V}$ such that $g : P \times \mathscr{V} \to P \times \mathscr{V}$ is given by $(p, \mathbf{v})g = (pg, g^{-1}\mathbf{v})$. Then $B = (P \times \mathscr{V})/G$ is the bundle space of the associated fibre bundle and we introduce the mapping $\pi': B \to M$ defined by $\pi'((p,\mathrm{v})G) = \pi(p)$. Thus the tangent bundle can be identified with the space of all pairs (m,t), where t is a tangent vector to M, so that

$$[(m, \mathbf{e}_1, \ldots \mathbf{e}_n), (v^1, \ldots v^n)]\, Gl(n, R) \to (m, v^i\, \mathbf{e}_i)$$

The tangent bundle is a special case of a vector bundle. This name is used when the structure group is a subgroup of the general linear group. The vector bundle can in turn be generalised to a tensor bundle in order to deal with higher-rank tensors by regarding $\mathscr{V}$ as a direct product of vector spaces and their duals. It can also be generalised to handle differential forms, in which case it is known as a Grassmann bundle. We will, in fact, be concerned with such a bundle.

Although the description we have been discussing so far is coordinate free, we wish to use a local coordinate description since this enables the formal connection with Yang-Mills fields to become more transparent. By using local coordinates, we will see that the structure group is defined by the allowable coordinate transformations on the base manifold. In this way we generate a bundle known as a coordinate bundle.

If $(x^1 \ldots x^n)$ are the local coordinates of point m in M, the local coordinates in P are $(x^i, X_j{}^i)$ where the n^2 functions $X_j{}^i$ define the n vectors of the frame $\mathbf{e}$ at x^i, i.e.

$$\mathbf{e}_j = X_j{}^i \frac{\partial}{\partial x^i}$$

If $g \in G$ with matrix representation $g_j{}^i$ then a new frame at m is given by (3.1) with

$$\mathbf{f}_j = \bar{X}_j{}^i \frac{\partial}{\partial x^i}$$

That is, in terms of local coordinates equation (3.1) becomes

$$\bar{X}_k{}^j = X_i{}^j g_k{}^i \tag{3.2}$$

Let us now look at equation (3.2) in another way. Let M be covered by a set of overlapping coordinate neighbourhoods and consider, in particular, two coordinate neighbourhoods U_α and U_β, such that $U_\alpha \cap U_\beta \neq 0$. At any point $m \in U_\alpha \cap U_\beta$ the coordinates of a point on the fibre is, say, $X^i_{(\alpha)j}$ in terms of the local coordinates U_α, while in terms of the local coordinates U_β, the same point on the fibre is $X^i_{(\beta)j}$. Thus equation (3.2) becomes

$$X^i_{(\beta)k} = X^i_{(\alpha)j}\, g^j_{(\alpha,\beta)k} \qquad \text{(No summation over } \alpha \text{ and } \beta) \tag{3.3}$$

which can be regarded as a 'coordinate transformation' on the fibre. It is now possible to introduce n linearly independent 1-forms $\theta^i_{(\alpha)}$ such that when $U_\alpha \cap U_\beta \neq 0$

$$\theta^i_{(\alpha)} = g^i_{(\alpha\beta)j}\, \theta^j_{(\beta)} \tag{3.4}$$

We can now form a global 1-form on the bundle space P

$$\omega^i = \theta^j_{(\alpha)} X^i_{(\alpha)j} = \theta^j_{(\beta)} X^i_{(\beta)j} \tag{3.5}$$

This global 1-form is defined at a point m of the manifold M, and we wish to find how the global 1-form changes at the point $m + dm$. To describe this change we introduce the displacement operator **d** which is called the exterior covariant derivative. Like the exterior derivative this is an anti-derivation so that we have

$$\mathbf{d}\omega^i = \mathbf{d}\theta^j X_j{}^i - \theta^j \mathbf{d}X_j{}^i \tag{3.5a}$$

Since the frames are a set of vectors, comparison of vectors at different points on a manifold is traditionally achieved through Levi-Civita parallelism, i.e. we write

$$\mathbf{d}X_j{}^i = dX_j{}^i + \Gamma^i_{mk}\, dx^m X_j{}^k = X_k{}^i \omega_j{}^k \tag{3.6}$$

where the Γ's are the usual Christoffel symbols while $\omega_k{}^i$ is the connection 1-form. In terms of the coordinate system $(x^i, \delta_j{}^i)$ we have

$$\omega_k{}^i = \Gamma^i_{mk}\, dx^m \tag{3.7}$$

As far as this paper is concerned it is sufficient to write

$$\mathbf{d}\theta^j = d\theta^j \tag{3.8}$$

As we will explain later, this is equivalent to assuming the existence of a fundamental horizontal 1-form sometimes called the solder 1-form (Bishop & Crittenden, 1964).

If we now substitute equations (3.8) and (3.6) into (3.5) we find

$$\mathbf{d}\omega^i = (d\theta^j - \theta^k \wedge \omega_k{}^j) X_j{}^i \tag{3.9}$$

The term in the brackets on the right-hand side of equation (3.9) is the usual expression for the torsion form.

To obtain the usual expression for the curvature form we write

$$\mathbf{d}(\mathbf{d}X_j{}^i) = \Omega'^i_k X_j{}^k \tag{3.10}$$

and it is not difficult to show that

$$\Omega'^i_j = d\omega_j{}^i - \omega_j{}^k \wedge \omega_k{}^i \tag{3.11}$$

Instead of introducing the connection 1-form $\omega_j{}^i$, we can introduce a left-invariant (under $g \in G$) 1-form, $\gamma_j{}^i$, defined by

$$\mathbf{d}X_j{}^i = \gamma_k{}^i X_j{}^k \tag{3.12}$$

By using equations (3.6), (3.10), and (3.12), we have

$$d\gamma_j{}^i + \gamma_j{}^k \wedge \gamma_k{}^i = Y_j{}^k \Omega'^n_k X_n{}^i = \Omega_j{}^i \tag{3.13}$$

where

$$Y_j{}^k X_k{}^i = \delta_j{}^i$$

Now in the fibre bundle theory the Ambrose-Singer theorem (1953) proves that the curvature form spans the Lie algebra of the restricted holonomy group H. Furthermore, this group is the group of isomorphisms of the fibre onto itself and hence is a sub-group of the structure group G. In other words, $\Omega_k'^n$ spans a sub-group of the structure group. Since $X_n{}^i$ can be generated from the coordinate system $(x^i, \delta_j{}^i)$ with the use of the matrix representations of G, $\Omega_j{}^i$ spans the adjoint representation of H. But H is only defined up to an automorphism so that $\Omega_j{}^i$ also spans the restricted holonomy group. Once again this can be made more obvious in a coordinate representation.

Since $\gamma_j{}^i$ is a left-invariant form we can expend it in terms of the set of r linearly independent left-invariant 1-forms, π^ρ, that span H, i.e.

$$\gamma_j{}^i = a^i_{\rho j}\,\pi^\rho \tag{3.14}$$

where the $a^i_{\rho j}$, are matrices with constant elements. Putting this into equation (3.13) we find

$$a^i_{\rho j}\,d\pi^\rho + a^k_{\rho j}\,\pi^\rho \wedge a^i_{\sigma k}\,\pi^\sigma = \Omega_j{}^i \tag{3.15}$$

The left-invariant 1-forms π^ρ are dual to a set of linearly independent vector fields Y_ρ defined through the relation

$$\pi^\sigma(Y_\rho) = \delta_\rho{}^\sigma \tag{3.16}$$

In terms of local coordinates, this duality operation is defined by writing

$$\mathbf{d}X_j{}^i\left(\frac{\partial}{\partial X_n{}^k}\right) = \delta_k{}^i\,\delta_j{}^n$$

By using equation (3.12) together with equation (3.14) we have

$$Y_\rho = a^i_{\rho k}\,X_j{}^k\,\frac{\partial}{\partial X_j{}^i} \tag{3.17}$$

since these vector fields span the Lie algebra of H, we have

$$[Y_\rho, Y_\sigma] = C^\tau_{\rho\sigma}\,Y_\tau \tag{3.18}$$

so that

$$[a^i_{\rho k}, a^j_{\sigma i}] = C^\tau_{\rho\sigma}\,a^j_{\tau k} \tag{3.19}$$

Thus the $a^i_{\rho k}$ span the Lie algebra of H. If we now combine equations (3.15) and (3.19) we find

$$a^i_{\tau j}[d\pi^\tau + \tfrac{1}{2}C^\tau_{\rho\sigma}\,\pi^\rho \wedge \pi^\sigma] = \Omega_j{}^i \tag{3.20}$$

Let us now introduce a set of quantities, B^τ, defined by

$$B^\tau = d\pi^\tau + \tfrac{1}{2}C^\tau_{\rho\sigma}\,\pi^\rho \wedge \pi^\sigma \tag{3.21}$$

so that

$$\Omega_j{}^i = a^i_{\tau j}\,B^\tau \tag{3.22}$$

Since the $a^{i}_{\rho j}$ form the basis of the Lie algebra of H, the $\Omega_j{}^i$, and hence $\Omega'_j{}^i$, also span the same Lie algebra.

In Section 4 we will show that the set of quantities B^τ have a very simple analogue in field theories, namely, they are just the Yang-Mills fields, written as 2-forms. For example, in the particular case in which the restricted holonomy group is an Abelian group, only one field is necessary and B^τ can then be identified with the electromagnetic field tensor written as a differential form.

Before establishing this connection with Yang-Mills fields, however, we wish to discuss a more general way of looking at connections and parallel displacements. In fact, we wish to consider these notions when generalised to the fibre bundle. We will see that we can introduce the notion of a horizontal lift which has a particularly simple meaning in the bundle and, in fact, can be identified with the generalised covariant derivatives that are introduced in the Yang-Mills theory.

3.2. *Connections, Parallel Displacement and Horizontal Lifts*

In Riemannian geometry parallel displacement is a method of relating tangent vectors at one point on a curve $\gamma(s)$ with tangent vectors at another point on the same curve. In fibre bundle theory, the corresponding notion is essentially a connection of a point on different fibres that lie along a curve $\gamma(s)$ in the base manifold M, i.e. a mapping of fibres onto fibres. In other words, the connection defines a curve in the bundle space which can be mapped onto $\gamma(s)$ in the base manifold. To make this notion clearer, we need to introduce a more general notion of connection which can be understood in the following way (see Bishop & Crittenden, 1964).

At each point p on the bundle space P there is a set of tangent vectors, t (i.e. tangent to the bundle space). In fact, these tangent vectors can be divided into two classes, horizontal and vertical. The vertical vectors are the tangent vectors which take us from one point on a fibre to a neighbouring point on the *same* fibre. Hence, there is an isomorphism of the Lie algebra of the structure group onto the vertical vectors. Formally the vertical vectors are those tangent vectors for which $d\pi(t)=0$; those tangent vectors for which $d\pi(t)\neq 0$ are called horizontal. These horizontal vectors connect neighbouring points of P in such a way that if one point is on one fibre, the neighbouring point is on a neighbouring fibre. Hence a set of horizontal vectors establishes a connection in the bundle. Once again, formally, a connection is a d-dimensional distribution, H, spanned by the horizontal vectors such that H is differential and, for every $p\in P$; $g\in G$, $dR_g H_p = H_{pg}$ which means that the set of horizontal vectors H_p at p are mapped onto the horizontal vectors H_{pg} at pg by means of a right translation dR_g of G.

The elements of H_p have the property that when they are projected via $d\pi$ onto the base manifold they become the vector fields in the tangent space of the base manifold. This mapping is one-to-one and hence, to every vector field on M, there corresponds a unique vector field $\tilde{X}$ on P called the *horizontal lift* of X with the property that $d\pi\tilde{X}_p = X_{\pi(p)}$ for every $p \in P$. It can also be shown that $dR_g\tilde{X} = \tilde{X}$ which means that the horizontal lift is invariant under the group G. Hence a connection can be described by a unique set of horizontal lifts which are invariant under G.

If a connection is defined in the above sense, then it is always possible to describe this connection in P by defining a unique 1-form ω on P. This form has values in the Lie algebra g of G such that ω is dual to the $\tilde{X}$ in the sense that $\omega(\tilde{X}) = 0$ if, and only if, $\tilde{X}$ is horizontal. This 1-form ω is the coordinate free description of the 1-form $\omega_k{}^i$ introduced in equation (3.6). The form ω is vertical. Any 1-form θ is called vertical if $\theta(X) = 0$, with X horizontal. Similarly any 1-form is called horizontal if $\theta(X) = 0$ with X vertical. It is in this sense that the coordinate description of θ used in equation (3.8) was called horizontal. In terms of the coordinate-free description $\mathbf{d} = d \circ h$ where h picks out the horizontal part of the vector field at $p \in P$ and d takes exterior derivative of it. Thus, since θ is horizontal, choose t to be horizontal and choose a coordinate representation such that

$$\theta(t) = \theta^i \tag{3.23}$$

then

$$\mathbf{d}\theta(t) = d\theta(ht) = d\theta(t) = d\theta^i \tag{3.24}$$

Let us now use the notion of a connection as described by the horizontal lifts to give a meaning to parallel displacement. Let $\gamma(s)$, with $0 \leqslant s \leqslant 1$, be a differential curve in the base space M along which we wish to establish the notion of parallel displacement. To do this we must lift $\gamma(s)$ onto the bundle space, P, so that, if for any point $p_0 \in P$ with $\pi(p_0) = \gamma(0)$, the lifted horizontal curve $\tilde{\gamma}(s)$ will give a unique point p_s in P with $\pi(p_s) = \gamma(s)$ for all s in the interval $[0, 1]$.

If p_0 is varied along the fibre $\pi^{-1}(\gamma(0))$, the corresponding lifted curve $\tilde{\gamma}(s)$ will give a mapping, τ, of the fibre $\pi^{-1}(\gamma(0))$ onto the fibre $\pi^{-1}(\gamma(1))$. We will call the mapping, τ, the parallel displacement along the curve $\gamma(s)$. Thus every lifted horizontal curve $\tilde{\gamma}(s)$ will indicate how to relate the tangent vectors to M along the curve $\gamma(s)$ in M. Since every lifted horizontal curve will be mapped into a lifted horizontal curve by R_g, the parallel displacement along any curve commutes with the action of G on P, i.e. $\tau \circ R_g = R_g \circ \tau$, $g \in G$.

The notion of a lifted curve corresponds, in fact, to the notion of the lift of a vector field. For if $\tilde{X}$ is the lift of a vector field X on M, then the integral curve of $\tilde{X}$ through the point $p_0 \in P$ is a lift of the integral curve of X through the point $\pi(p_0)$. Thus a set of horizontal lifts describe parallel translation in the fibre bundle and is, in fact, a generalisation of the notion of a covariant derivative.

If we extend our discussion of parallel displacement to consider closed curves or loops in a region of the manifold M which is simply connected (i.e. all loops in the region are homotopic to zero), then we make contact with the restricted holonomy group already referred to in the previous section.

Finally, let us look at the form of the horizontal lifts in terms of a local coordinate description. If we choose the coordinates $(x^i, \delta_j{}^i)$ we can write

$$\pi^\rho = \theta_k{}^\rho\, dx^k \tag{3.25}$$

so that from equation (3.6), (3.12) and (3.7) we find

$$\gamma_j{}^i = a^i_{\rho j}\, \theta_k{}^\rho\, dx^k \equiv \Gamma^i_{kj}\, dx^k \tag{3.26}$$

Our vector field X_i on the base manifold can also be written as $X_i = \partial/\partial x^i$.

In terms of the local coordinates $(x^i, X_j{}^i)$ we have

$$\gamma_j{}^i = Y_j{}^m(dX_m{}^i + a^i_{\rho n}\, \theta_k{}^\rho\, dx^k\, X_m{}^n) \tag{3.27}$$

The horizontal lift, $\tilde{X}$, of the vector field X_i is obtained from

$$\gamma(\tilde{X}) = 0 \qquad \text{or} \qquad \gamma_j{}^i(\tilde{X}_k) = 0 \tag{3.28}$$

If we write

$$\tilde{X}_k = \frac{\partial}{\partial x^k} + B^i_{kj} \frac{\partial}{\partial X_j{}^i} \tag{3.29}$$

it is not difficult to show that from equation (3.5) and (3.6) we obtain

$$B^i_{kj} = -a^i_{\rho n}\, \theta_k{}^\rho\, X_j{}^n \tag{3.30}$$

so that equation (3.29) becomes

$$\tilde{X}_k = \frac{\partial}{\partial x^k} - a^i_{\rho n}\, \theta_k{}^\rho\, X_j{}^n \frac{\partial}{\partial X_j{}^i} \tag{3.31}$$

If we now compare this with equation (3.17), we can finally write the horizontal lift as

$$\tilde{X}_k = \frac{\partial}{\partial x^k} - \theta_k{}^\rho\, Y_\rho \tag{3.32}$$

where the Y_ρ span the holonomy group H.

In the next section we will briefly discuss the Yang-Mills theory and we will see that equation (3.32) can be identified with the generalised covariant derivative introduced in this theory.

4. *Yang-Mills Fields and Fibre Bundles*

We now wish to show the explicit connection between Yang-Mills fields and fibre bundle theory. Firstly, let us recall the basic physical notions behind the Yang-Mills theory by discussing the particular case of the proton-neutron system which can be regarded as different isospin states of the same particle, the nucleon. In the absence of an electromagnetic field the orientation of the 3-axis in isospace has no direct physical significance and it is a matter of convention as to which state is called the proton.

Provided only one point in space-time is considered, this convention can lead to no difficulties, but if two or more separate space-time points are considered, the relative orientation of the 3-axis at each point is not arbitrary, i.e. we must have a consistent description at all points. This implies that we have a kind of non-locality to consider. However, since this is inconsistent with the notion of local field theories, Yang and Mills considered the possibility of introducing new fields in such a way as to make the theory invariant under independent rotations in isospace at every point in space-time.

Let us now look at these physical ideas from the fibre bundle point of view. Consider the isospace as a fibre over each point of the space-time manifold. The 'rotations' in isospace which, in this case, are elements of the $SU(2)$ group, map the fibre onto itself. The gauge groups, $SU(2)$, can be regarded as the structure group with space-time as the base manifold. Thus there is a formal correspondence between the fibre bundle and the Yang-Mills theory (Lubkin, 1963; Herman, 1966).

The Yang-Mills fields have been thought of as introducing an 'interaction' between particles at different points in space-time. In fact, this was the way in which Ne'eman (1961) was led to postulate the existence of vector mesons with $SU(3)$ properties. However, in terms of the basic notions that we are using, it is not meaningful to talk in terms of 'particles in interaction'. In one sense we are closer to the Einstein notion that forces, or interactions, manifest themselves through some invariant features of a geometry; for example, relativity uses the curvature properties. This notion could be extended to the Yang-Mills case where 'interaction' is now described by the curvature properties of the bundle. However, this interpretation suffers from one serious difficulty. Until now the structure group has been regarded as a pure gauge transformation in the sense that it does not arise from the properties of the base manifold (i.e. space-time) so that there is, at present, no connection between the internal properties and the external properties. This is a feature that we wish to avoid.

Before discussing this point in detail let us first look further into the formal analogy between the fibre bundle and the Yang-Mills theory and show in what way the physical phenomena can be discussed in terms of the curvature properties of the bundle. The feature we will discuss is independent of whether the structure group is a pure gauge transformation or not.

We have shown that the curvature properties of the bundle arise from a consideration of the parallel displacement of fibres along curves in a suitable base manifold. Parallel displacement is described by means of the horizontal lift and the curvature properties are directly related to the quantities B^{τ} (equation 3.21). These quantities should, therefore, have analogues in the Yang-Mills theory. They have in fact and to show this in detail, let us consider again the local coordinate description. The Yang-Mills formalism requires a Lagrangian $L(\psi, \partial_{\mu}\psi)$ which is to be invariant under the general gauge transformation

$$\psi \to \exp\left[iY_{\rho}\,\epsilon\,(x)\right]\psi \tag{4.1}$$

where Y_ρ are a set of non-commuting operators that span the Lie algebra of the gauge group. This Lagrangian is made gauge invariant by introducing a compensating or Yang-Mills potential $\theta_\mu{}^\rho$ which is used to replace the partial derivative, ∂_μ, by a gauge invariant derivative

$$\nabla_\mu = \frac{\partial}{\partial x^\mu} - \theta_\mu{}^\rho Y_\rho \tag{4.2}$$

(see for example, Utiyama (1956), equation (1.10)). If we now compare this equation with (3.32), we see that it is identical with the horizontal lift of the vector fields $X_\mu = \partial/\partial x^\mu$. Thus the gauge invariant derivative is, in fact, a generalised covariant derivative in the bundle.

If we form the commutator for ∇_μ we find

$$[\nabla_\nu, \nabla_\mu] = \left[\frac{\partial \theta_\nu{}^\tau}{\partial x^\mu} - \frac{\partial \theta_\mu{}^\tau}{\partial x^\nu} - \tfrac{1}{2} C^\tau_{\rho\sigma}(\theta_\mu{}^\rho \theta_\nu{}^\sigma - \theta_\nu{}^\rho \theta_\mu{}^\sigma)\right] Y_\tau \equiv B^\tau_{\mu\nu} Y_\tau \tag{4.3}$$

where we have used equation (3.18). We see that the $B^\tau_{\mu\nu}$ are the Yang-Mills fields derived from the potentials $\theta_\mu{}^\tau$ (see Utiyama (1956) equation (1.18)). We can now write the Yang-Mills fields as a 2-form in the following manner

$$B^\tau = \sum_{\mu<\nu} B^\tau_{\mu\nu}\, dx^\mu \wedge dx^\nu \tag{4.4}$$

This is just the expression (3.21), for if we write $\pi^\rho = \theta_\mu{}^\rho dx^\mu$ in equation (2.21) we can easily show that

$$B^\tau_{\mu\nu} = \frac{\partial \theta_\nu{}^\tau}{\partial x^\mu} - \frac{\partial \theta_\mu{}^\tau}{\partial x^\nu} - \tfrac{1}{2} C^\tau_{\rho\sigma}(\theta_\mu{}^\rho \theta_\nu{}^\sigma - \theta_\nu{}^\rho \theta_\mu{}^\sigma) \tag{4.5}$$

which establishes the formal connection between the Yang-Mills description and the fibre bundle approach.

If we adopt the more conventional view that field theory is to be regarded as the basic theory, then the connection between the fibre bundle and the Yang-Mills theory does not add any new physical content and is to be regarded as nothing more than a powerful mathematical tool with which to study the theory. However, we have already indicated the inadequacy of the field theoretic approach (Hiley, 1968; Bohm *et al.*, 1970) and we are suggesting a different approach in which activity and movement are regarded as basic. These notions require new forms of description and we have suggested that the cellular (simplicial complex) description seems appropriate. In this theory, the phenomena and the apparatus are not to be regarded as disjoint entities in interaction, but are to be incorporated into a total structure in which analysis into parts is not relevant. Thus our theory will require a description of this structure. In the approximation that was introduced in Section 2, we replaced the overall cell structure by a frame bundle description, that is, the frame bundle is a natural description of the overall structure to this level of approximation and, therefore, we can begin to ask questions about this overall structure without having to consider fields in a fundamental way. Thus, from our point of view, the fibre bundle approach can add new physical content to the theory. For example, in the next section we show how certain aspects of the overall structure are characterised by a set of 2-forms and their duals, together with a set of generalised currents. These forms enable us to make inferences about the structure of the underlying cell complex.

5. *Frame Bundles and Current Algebras*

In this section we wish to give a structural interpretation to the currents that arise in the Yang-Mills theory by considering how analogous quantities arise in fibre bundle theory. Firstly, let us recall how the electromagnetic field equations arise in the Yang-Mills theory. The appropriate gauge group is a one parameter Abelian group and, therefore, only one potential is needed, i.e. $\theta_\mu{}^\tau = A_\mu$ the usual four vector potential. Since $C^\tau_{\rho\sigma}$ is zero for this group, equation (4.5) becomes

$$B_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \tag{5.1}$$

so that $B_{\mu\nu}$ is simply the electromagnetic field tensor. Hence from (4.4), the electromagnetic 2-form is

$$f = B_{\mu\nu}\, dx^\mu \wedge dx^\nu \tag{5.2}$$

while the potential vector is written as

$$\pi = A_\mu\, dx^\mu \tag{5.3}$$

Substituting equations (5.2) and (5.3) into (3.21) we get a pair of Maxwell's equations

$$f = d\pi \tag{5.4}$$

or by taking the exterior derivative on both sides, we have

$$df = 0 \tag{5.5}$$

The remaining pair of Maxwell's equations require the introduction of the dual 2-form $*f$ (see Bohm *et al.*, 1970) and this is related to the current via

$$d{*f} = j \tag{5.6}$$

where j is the current 3-form $j = \epsilon_{\alpha\beta\gamma s}\, j^s\, dx^\alpha\, dx^\beta\, dx^\gamma$. By taking the exterior derivative of (5.6) we obtain the conservation of current equation

$$dj = 0 \tag{5.7}$$

Now in equation (3.22) we have shown that the B^τ are in fact related to the curvature forms $\Omega_j{}^i$. Furthermore, Maxwell's equations show that $*B^\tau$, and hence the dual curvature form $*\Omega_j{}^i$ are also needed to specify the physically relevant features of the bundle. We therefore assume that when the structure group is non-Abelian both B^τ and $*B^\tau$ (and hence $\Omega_j{}^i$ and $*\Omega_j{}^i$) are again needed, but in this case $dB^\tau \neq 0$ indicating that we are dealing with what is regarded, from the conventional point of view, as a non-linear effect. In analogy with the electromagnetic case, we define a generalised current, j^τ, by

$$d{*B^\tau} = j^\tau \tag{5.8}$$

Since the exterior derivative is nilpotent, the currents are conserved quantities.

We can now relate the currents to the dual curvature forms through equation (3.22). Since the $a^i_{\tau j}$ are constants we have

$$d{*\Omega_j{}^i} = a^i_{\tau j}\, d{*B^\tau} \tag{5.9}$$

From equation (5.8) we have

$$d{*\Omega_j{}^i} = a^i_{\tau j}\, j^\tau \tag{5.10}$$

Hence we have a current form

$$J_j{}^i = a^i_{\tau j}\, j^\tau \tag{5.11}$$

with the currents $J_j{}^i$ spanning the Lie algebra of the holonomy group which is a sub-group of the structure group. Thus we have a set of current commutation relations which arise naturally in the description we are using.

One connection between currents and the holonomy group has already been pointed out by Lubkin (1963) and Loos (1964, 1966). However, their proposals differ from ours in several essential ways. They regard the Lie algebra spanned by the currents as arising from a pure gauge group; the corresponding holonomy group is then interpreted as describing the structure of an 'internal' space. The base manifold that they use is the space-time manifold which itself gives rise to a structure group describing the 'external' properties. This implies an artificial division between the 'internal' and 'external' properties. To remove this division it is necessary to consider a larger structure from which these two aspects emerge. Unfortunately, unification along these lines sooner or later leads to the difficulties pointed out by O'Raifeartaigh (1965).

Our basic concepts, however, are different from the conventional ones and, as a consequence, the O'Raifeartaigh theorem is not applicable. Indeed, we hope to avoid the difficulties even in the approximation in which the cell structure is replaced by the frame bundle. Our base manifold is a phase space and we do not regard the structure group as a pure gauge transformation, but rather as induced naturally by allowable coordinate transformations in phase space itself. As space-time is already contained implicitly in phase-space, it is necessary for us to reduce (rather than to extend) the bundle structure in an appropriate way.

We do not wish to discuss the precise nature of the possible structure groups that arise from allowable coordinate transformations in phase space in this paper. Clearly, they are related to the canonical transformations in some way. However, unitary symmetry being a sub-group of the canonical transformations, cannot be explained without further assumptions of a physical nature. These assumptions will be discussed in a later paper.

Our discussion of the frame bundle has made full use of differential forms. This feature is very important from our point of view as we can regard these forms as defining a de Rahm cohomology which has values in the reals, that is, our description uses the reals in an essential way. On the other hand, experiment suggests that it is the integers which play an important role (e.g. energy levels, integral charges, etc.) and hence in any alternative to quantum theory, the integers must play a fundamental role. We have already suggested elsewhere (Hiley, 1968; Bohm *et al.*, 1970) that one way to introduce this kind of discreteness is to consider cohomologies with values in the integers. In fact, we have demonstrated how it is possible to obtain a description of integral charges through a re-interpretation of Maxwell's electromagnetic theory. This was achieved by assuming that the de Rahm cohomology reflects a more basic cohomology which has values in the integers. More precisely, we propose that when the integral values in the basic cohomology are large, the de Rahm cohomology gives an adequate approximation to the basic discrete structure. In other words, we are suggesting a kind of correspondence principle which has features that are in some ways similar to the one introduced by Bohr.

One can argue that as a result of the similarity of the two cohomologies, the various results obtained from the frame bundle can be given a discrete structural meaning. For example, one way would be to assume that the symmetry properties of the cells themselves are again described by the structure group which, in this case, is discrete. A form of holonomy group could then be used, but it seems inappropriate to regard it as describing curvature properties since curvature has no meaning in a discrete structure. Formally, the group arises from a consideration of how neighbouring cells fit together, that is, how they cohere. Thus we suggest that in this case the group describes the coherence properties of the cells and could, perhaps, more appropriately be called the coherence group. However, as the work of Schild (1949) shows, it may be inadequate to regard the symmetry properties and the coherence properties as being described by a group. Some more general notion will be needed and we are at present looking into various possibilities. However, even in the case that we have considered, the generalised currents can still be given a discrete structural meaning in terms of the boundaries of structures described by coherence ('curvature') forms (see, for example, Bohm *et al.*, 1970). In the discrete case, the currents do not span a Lie algebra. It can be shown that a Lie algebra only emerges as a result of an approximation and hence, from our point of view, the Lie algebra description is to be regarded as an approximation. It is not clear at this stage whether, as a result of this, the discrete case will contain features similar to those used in 'broken' symmetries. However, what is clear is that in our description the charges associated with the currents will be discrete and this is precisely what is required by experiment.

6. *Conclusion*

We have continued to investigate some aspects of a new theory suggested in a previous paper (Bohm *et al.*, 1970). There we questioned the basic concepts of conventional theories and argued that radical changes may be needed to give a clearer understanding of quantum phenomena. To discuss these new concepts new mathematical descriptions are needed and we suggested that they should be based on a cellular structure rather than the continuum. In this paper we have pointed out that the bilocal structure of Yukawa and its extension by Takabayasi can be considered as imposing a cellular structure in space-time. As yet there is no evidence for such a structure in space-time itself, but quantum theory indicates that there is a cellular structure in phase space, and we have discussed some of the implications of such a structure. We have then shown that some aspects of this cell structure can be described approximately by the mathematical theory of fibre bundles, in particular, the frame bundle. This bundle is shown to contain features which already appear in the Yang-Mills theory even though the two approaches start from very different standpoints. By a comparison of these two approaches, we are able to give a new meaning to gauge transformations. They are no longer associated with the arbitrariness of the phase of the wave-function, but directly related to the symmetry properties of the basic cells in the description. In turn, these symmetry properties are intimately related to the overall experimental conditions, indicating that the bundle description contains a kind of wholeness suggested by quantum theory.

Although we show the similarity between the fibre bundle and Yang-Mills theory, we do not regard the quantities $\theta_\mu{}^\tau$ as giving rise to 'interactions'. In our theory we do not have separately existing systems except in some suitable approximation. The phenomena arise from the whole movement and the structure is the phenomena. In other words, the interaction is incorporated in the structure in a way that is, in principle, similar to general relativity. We suggest that this type of approach is needed in high energy physics because in some cases the interactions are so strong that it is impossible, even in principle, to talk of two separately existing systems in interaction.

To describe the overall structure of the bundle, we need a curvature form and its dual, together with a set of conserved currents which span the Lie algebra of the holonomy group. Thus the description of the structure naturally requires a set of current commutation relations. We do not discuss which Lie group arises naturally in our structure in this paper, but we will show what additional assumptions are required to make contact with unitary symmetry elsewhere.

If, instead of using the approximate fibre bundle description, we make full use of the cellular description by assuming the structure to be an abstract simplicial complex, many of the relations can be given a meaning in the manner discussed in Bohm *et al.* (1970). Although there are points which require clarification, we find that it is still possible to introduce currents, but in this case they will no longer span a Lie algebra except in some limit. Thus, in our view, a description using Lie algebras is only an approximation to a more fundamental theory having discreteness as an essential feature.

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